

Partially superintegrable systems on Poisson manifolds

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Abstract

Superintegrable (non-commutative completely integrable) systems on a symplectic manifold conventionally are considered. However, their definition implies a rather restrictive condition $2n = k + m$ where $2n$ is a dimension of a symplectic manifold, k is a dimension of a pointwise Lie algebra of a superintegrable system, and m is its corank. To solve this problem, we aim to consider partially superintegrable systems on Poisson manifolds where $k + m$ is the rank of a compatible Poisson structure. The according extensions of the Mishchenko–Fomenko theorem on generalized action-angle coordinates is formulated.

1 Introduction

The Liouville–Arnold theorem for completely integrable systems [2, 18], the Poincaré–Lyapounov–Nekhoroshev theorem for commutative partially integrable systems [7, 21] and the Mishchenko–Fomenko theorem (Theorem 2.4) for the superintegrable ones [3, 4, 20] on symplectic manifolds state the existence of action-angle coordinates around a compact invariant submanifold of an integrable system which is a torus T^m . These theorems have been extended to a general case of invariant submanifolds which need not be compact, but are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r, \quad T^r = \overset{r}{\times} S^1, \quad (1.1)$$

(Theorems 2.5, 2.6 and 2.3, respectively) [6, 10, 12, 25, 26, 30].

However, Definition 2.1 of a superintegrable (non-commutative completely integrable) system on a symplectic manifold is rather restrictive. Its item (iii) requires that the matrix function s_{ij} (2.3) must be of corank $m = 2n - k$ (2.4) where $2n$ is a dimension of a symplectic manifold and k is a number of independent generating functions of a superintegrable system. In particular, commutative partially integrable systems on a symplectic manifold fail to satisfy this condition (Remark 2.1).

Example 1.1: The Kepler problem on an $(n = 2)$ -dimensional configuration space $\mathbb{R}^2 \setminus \{0\}$ possesses three integrals of motion: an orbital momentum M_{12}

and two components of the Runge–Lenz vector A^a [25, 26]. They constitute two different superintegrable systems: (i) with a Lie algebra $so(3)$ on a domain U_- of a phase space $\mathbb{R}^4 \setminus \{0\}$ of negative energy, and (ii) with a Lie algebra $so(2, 1)$ on a domain $U_+ \subset \mathbb{R}^4 \setminus \{0\}$ of positive energy. However, if $n > 2$, a number of integrals of motion (M_{ij}, A^i) , $i = 1, \dots, n$, of the Kepler system is more than $2n$, and they fail to form any superintegrable system. In this case, one however can consider a partially superintegrable system with the generating functions (M_{12}, A^1, A^2) because orbits of a motion of a Kepler system is well known to lie in a plane. \square

To avoid the restriction $m = 2n - k$ (2.4), we aim to consider partially superintegrable systems on Poisson manifolds (Definition 5.2). The following are their examples.

Example 1.2: Let $F = (F_1, \dots, F_k)$ be a superintegrable system of corank $2n - k$ on a $2n$ -dimensional connected symplectic manifold (Z, Ω) . Let X be an r -dimensional manifold regarded as a Poisson manifold with a zero Poisson structure (Example 6.3). A manifold product $Z \times Q$ can be endowed with a product Poisson structure w of rank $2n$ (Example 6.4). Then the pull-back of functions F_i onto $Z \times X$ exemplifies a partially superintegrable system of corank $m = 2n - k$ where $2n$ is a rank of a Poisson structure on a $(2n + r)$ -dimensional Poisson manifold (Definition 5.2). \square

Example 1.3: Commutative partially integrable systems on Poisson manifolds (Definition 4.2) exemplify partially superintegrable systems. \square

A key point is that invariant submanifolds of a superintegrable system are integral manifolds of a certain commutative partially integrable system on a symplectic manifold (Remark 2.2). As a consequence, the proof of above mentioned generalized Mishchenko–Fomenko theorem (Theorem 2.3) for superintegrable systems is reduced to generalized Poincaré–Lyapounov–Nekhoroshev Theorem 2.6 for commutative partially integrable systems on a symplectic manifold (Section 2). Therefore, we start our investigation of partially superintegrable systems with commutative partially integrable systems on Poisson manifolds (Section 4) [5, 10].

Our goal is that, in a case of partially integrable systems on Poisson manifolds, the above mentioned restriction condition (2.4) comes to a form $k + m = r$ where r is the rank of a Poisson structure on a manifold Z , but not a dimension of Z (Lemma 5.1).

The extended Mishchenko–Fomenko theorem on generalized action-angle coordinates in the case of symplectic superintegrable systems (Theorem 2.3) is extended to partially superintegrable systems on Poisson manifolds (Theorem 5.2). Its proof also is reduced to Theorems 4.2 and 4.5 for commutative partially integrable systems on Poisson manifolds.

2 Superintegrable systems on symplectic manifolds

Throughout the work, all functions and maps are smooth, and manifolds are finite-dimensional smooth real and paracompact.

DEFINITION 2.1: Let (Z, Ω) be a $2n$ -dimensional connected symplectic manifold, and let $(C^\infty(Z), \{, \})$ be a Poisson algebra of smooth real functions on Z . A subset

$$F = (F_1, \dots, F_k), \quad n \leq k < 2n, \quad (2.1)$$

of a Poisson algebra $C^\infty(Z)$ is called a superintegrable system if the following conditions hold.

(i) All the functions F_i (called the generating functions of a superintegrable system) are independent, i.e., a k -form $\bigwedge^k dF_i$ nowhere vanishes on Z . It follows that a surjection

$$\widehat{F} : Z \rightarrow N = \times_i F_i(Z) \subset \mathbb{R}^k \quad (2.2)$$

is a submersion, i.e., a fibred manifold over a domain (contractible open subset) $N \subset \mathbb{R}^k$ endowed with the coordinates (x_i) such that $x_i \circ \widehat{F} = F_i$. Fibres of the fibred manifold \widehat{F} (2.2) are called the invariant submanifolds of a superintegrable system.

(ii) There exist smooth real functions s_{ij} on N such that

$$\{F_i, F_j\} = s_{ij} \circ \widehat{F}, \quad i, j = 1, \dots, k. \quad (2.3)$$

(iii) The matrix function \mathbf{s} with the entries s_{ij} (2.3) is of constant corank

$$m = 2n - k, \quad 2n = \dim Z, \quad k = \dim N, \quad (2.4)$$

at all points of N . \square

If $k > n$, the matrix \mathbf{s} is necessarily non-zero. If $k = n$, then $\mathbf{s} = 0$, and we are in the case of completely integrable systems as follows.

DEFINITION 2.2: The subset (F_1, \dots, F_n) (2.1) of a Poisson algebra $C^\infty(Z)$ on a symplectic manifold (Z, Ω) is called the completely integrable system if F_i are independent functions in involution. \square

Therefore, superintegrable systems sometimes are called non-commutative completely integrable systems. However, this notion differs from that in [17].

Remark 2.1: A family $\{S_1, \dots, S_m\}$ of $m \leq n$ independent smooth real functions in involution on a symplectic manifold (Z, Ω) is called the (commutative) partially integrable system. It should be emphasized that a commutative partially integrable system on a symplectic manifold fails to be a particular superintegrable system because the condition (2.4) in item (iii) of Definition 2.1 is not satisfied, unless $m = n$ and it is a completely integrable system. \square

The following two assertions clarify the structure of superintegrable systems [4, 6, 12, 26].

LEMMA 2.1: Given a symplectic manifold (Z, Ω) , let $\pi : Z \rightarrow N$ be a fibred manifold such that, for any two functions f, f' constant on fibres of π , their Poisson bracket $\{f, f'\}$ also is well. By virtue of Theorem 6.6, N is provided with a unique coinduced Poisson structure $\{, \}_N$ such that π is a Poisson morphism. \square

Since any function constant on fibres of π is a pull-back of some function on N , the superintegrable system (2.1) with $\pi = \widehat{F}$ satisfies the condition of Lemma 2.1 due to item (i) of Definition 2.1. Thus, a base N of the fibred manifold \widehat{F} (2.2) is endowed with a coinduced Poisson structure of corank m . With respect to coordinates x_i in item (i) of Definition 2.1, its bivector field reads

$$w = s_{ij}(x_k) \partial^i \wedge \partial^j. \quad (2.5)$$

LEMMA 2.2: Given a fibred manifold $\pi : Z \rightarrow N$ with connected fibres in Lemma 2.1, the following conditions are equivalent [4, 19]:

- (i) the corank of the coinduced Poisson structure $\{, \}_N$ on N equals $m = \dim Z - \dim N$,
- (ii) the fibres of π are isotropic,
- (iii) the fibres of π are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back π^*C of Casimir functions C of a coinduced Poisson structure on N . \square

It is readily observed that the fibred manifold \widehat{F} (2.2) obeys condition (i) of Lemma 2.2 due to item (iii) of Definition 2.1, namely, $k - m = 2(k - n)$.

Remark 2.2: The pull-back π^*C of Casimir functions in item (iii) of Lemma 2.2 are in involution with all functions on Z which are constant on fibres of π . Let N admit a family of m independent Casimir functions (C_λ) . Then their pull-back π^*C_λ constitute a commutative partially integrable system on a symplectic manifold (Z, Ω) (Remark 2.1). In this case it follows from item (iii) of Lemma 2.2 that invariant submanifolds of a superintegrable system are integral manifolds of this commutative partially integrable system. As a consequence, the proof of generalized Mishchenko–Fomenko Theorem 2.3 is reduced to the generalized Poincaré–Lyapounov–Nekhoroshev theorem (Theorem 2.6) for commutative partially integrable systems on a symplectic manifold. \square

Remark 2.3: In many applications, condition (i) of Definition 2.1 fails to hold. It can be replaced with that a subset $Z_R \subset Z$ of regular points (where $\bigwedge^k dF_i \neq 0$) is open and dense. Let M be an invariant submanifold through a regular point $z \in Z_R \subset Z$. Then it is regular, i.e., $M \subset Z_R$. Let M admit a regular open saturated neighborhood U_M (i.e., a fibre of \widehat{F} through a point of U_M belongs to U_M). For instance, any compact invariant submanifold M has such a neighborhood U_M . The restriction of functions F_i to U_M defines a

superintegrable system on U_M which obeys Definition 2.1. In this case, one says that a superintegrable system is considered around its invariant submanifold M . We refer to [25, 26] for a global analysis of superintegrable systems. \square

Given a superintegrable system in accordance with Definition 2.1, the above mentioned generalization of the Mishchenko – Fomenko theorem to non-compact invariant submanifolds states the following [6, 12, 25, 26].

THEOREM 2.3: Let the Hamiltonian vector fields ϑ_i (6.13) of the generating functions F_i be complete, and let fibres of the fibred manifold \hat{F} (2.2) be connected and mutually diffeomorphic. Then the following hold.

(I) The fibres of the fibred manifold \hat{F} (2.2) are diffeomorphic to the toroidal cylinder (1.1) where $m = 2n - k$.

(II) Given a fibre M of the fibration \hat{F} (2.2), there exists its open saturated neighborhood U_M which is a trivial principal bundle

$$U_M = N_M \times (\mathbb{R}^{m-r} \times T^r) \xrightarrow{\hat{F}} N_M \quad (2.6)$$

with the structure additive group (1.1).

(III) A neighborhood U_M is provided with the bundle (generalized action-angle) coordinates $(I_\lambda, q^A, y^\lambda)$, $\lambda = 1, \dots, m$, $A = 1, \dots, 2(n - m)$, such that: (i) the generalized angle coordinates (y^λ) are coordinates on a toroidal cylinder, i.e., fibre coordinates on the fibre bundle (2.6); (ii) the (I_λ, q^A) are coordinates on its base N_M where the action coordinates (I_λ) are values of independent Casimir functions of the coinduced Poisson structure $\{, \}_N$ on N_M ; and (iii) a symplectic form Ω on U_M reads

$$\Omega = dI_\lambda \wedge dy^\lambda + \Omega_{AB}(I_\beta, q^C) dq^A \wedge dq^B.$$

\square

Outline of proof: It follows from item (iii) of Lemma 2.2 that every fibre M of the fibred manifold (2.2) is a maximal integral manifold of an involutive distribution spanned by the Hamiltonian vector fields v_λ of the pull-back \hat{F}^*C_λ of m independent Casimir functions $\{C_1, \dots, C_m\}$ of the coinduced Poisson structure $\{, \}_N$ (2.5) on an open neighborhood N_M of a point $\hat{F}(M) \in N$. Let us put $U_M = \hat{F}^{-1}(N_M)$. It is an open saturated neighborhood of M . Consequently, invariant submanifolds of a superintegrable system (2.1) on U_M are integral manifolds of a commutative partially integrable system

$$S = (\hat{F}^*C_1, \dots, \hat{F}^*C_m), \quad 0 < m \leq n, \quad (2.7)$$

on a symplectic manifold (U_M, Ω) (Remark 2.2). Therefore, statements (I) – (III) of Theorem 2.3 are the corollaries of Theorem 2.6 below. Its condition (i) is satisfied as follows. Let M' be an arbitrary fibre of the fibred manifold $\hat{F} : U_M \rightarrow N_M$ (2.2). Since

$$\hat{F}^*C_\lambda(z) = (C_\lambda \circ \hat{F})(z) = C_\lambda(F_i(z)), \quad z \in M',$$

the Hamiltonian vector fields v_λ on M' are \mathbb{R} -linear combinations of Hamiltonian vector fields ϑ_i of generating functions F_i . It follows that v_λ on M' are elements of a finite-dimensional real Lie algebra of vector fields on M' generated by vector fields ϑ_i . Since vector fields ϑ_i are complete, the vector fields v_λ on M' also are well (Remark 2.4 below). Consequently, these vector fields are complete on U_M because they are vertical vector fields on $U_M \rightarrow N$. \square

Remark 2.4: If complete vector fields on a smooth manifold constitute a basis for a finite-dimensional real Lie algebra, any element of this Lie algebra is complete [23]. \square

Remark 2.5: The condition of the completeness of Hamiltonian vector fields of generating functions F_i in Theorem 2.3 is rather restrictive. One can replace it with that the Hamiltonian vector fields v of the pull-back onto Z of Casimir functions on N are complete. \square

If the conditions of Theorem 2.3 are replaced with that fibres of the fibred manifold \widehat{F} (2.2) are compact and connected, this theorem restarts the Mishchenko–Fomenko one as follows [3, 4, 20].

THEOREM 2.4: Let fibres of the fibred manifold \widehat{F} (2.2) be compact and connected. Then they are diffeomorphic to a torus T^m , and statements (II) – (III) of Theorem 2.3 hold. \square

Remark 2.6: In Theorem 2.4, the Hamiltonian vector fields v_λ are complete because fibres of the fibred manifold \widehat{F} (2.2) are compact. As well known, any vector field on a compact manifold is complete. \square

If F (2.1) is a completely integrable system, the coinduced Poisson structure on N equals zero, and the generating functions F_i are the pull-back of n independent functions on N . Then Theorems 2.4 and 2.3 come to the well-known Liouville–Arnold theorem [2, 18] and its generalization (Theorem 2.5 below) to the case of non-compact invariant submanifolds [5, 12, 26], respectively.

THEOREM 2.5: Given a completely integrable system F in accordance with Definition 2.2, let the Hamiltonian vector fields ϑ_i of functions F_i be complete, and let fibres of the fibred manifold \widehat{F} (2.2) be connected and mutually diffeomorphic. Then items (I) and (II) of Theorem 2.3 hold, and its item (III) is replaced with the following one.

(III') The neighborhood U_M (2.6) where $m = n$ is provided with the bundle (generalized action-angle) coordinates (I_λ, y^λ) , $\lambda = 1, \dots, n$, such that the angle coordinates (y^λ) are coordinates on a toroidal cylinder, and the symplectic form Ω on U_M reads

$$\Omega = dI_\lambda \wedge dy^\lambda. \quad (2.8)$$

\square

In a general setting, one considers commutative partially integrable systems on a symplectic manifold (Remark 2.1). The Poincaré–Lyapounov–Nekhoroshev

theorem [7, 8, 21] generalizes the Liouville–Arnold one to a commutative partially integrable system with compact invariant submanifolds. Forthcoming Theorem 2.6 is concerned with a generic commutative partially integrable system on a symplectic manifold [10, 25, 26].

Given a commutative partially integrable system $S = \{S_1, \dots, S_m\}$ of $m \leq n$ independent smooth real functions in involution on a symplectic manifold (Z, Ω) , we have a fibred manifold

$$\widehat{S} : Z \rightarrow X = \times^m S_\lambda(Z) \subset \mathbb{R}^m \quad (2.9)$$

over a domain $X \subset \mathbb{R}^m$. We agree to call its fibres the invariant submanifolds of a commutative partially integrable system though it is not a superintegrable system (Remark 2.1), unless $m = n$ and it is a completely integrable system.

Hamiltonian vector fields v_λ of generating functions S_λ are mutually commutative and independent. Consequently, they span an m -dimensional involutive distribution on Z whose maximal integral manifolds constitute an isotropic foliation \mathcal{S} of Z . Its leaves are called the integral manifolds of a commutative partially integrable system. Because functions S_λ are constant on leaves of this foliation, each fibre of the fibred manifold \widehat{S} (2.9) is foliated by the leaves of a foliation \mathcal{S} .

If $m = n$, we are in the case of a completely integrable system, and its integral manifolds (i.e., leaves of \mathcal{S}) are connected components of its invariant submanifolds (i.e., fibres of the fibred manifold \widehat{S} (2.9)).

THEOREM 2.6: Let a commutative partially integrable system $S = \{S_1, \dots, S_m\}$ on a symplectic manifold (Z, Ω) satisfy the following conditions.

- (i) The Hamiltonian vector fields v_λ of S_λ are complete.
- (ii) The foliation \mathcal{S} is a fibred manifold $\pi_S : Z \rightarrow N$ whose fibres are mutually diffeomorphic and, being integral manifolds, are connected.

Then the following hold [10, 12, 25, 26].

- (I) The fibres of \mathcal{S} are diffeomorphic to the toroidal cylinder (1.1).
- (II) Given a fibre M of \mathcal{S} , there exists its open saturated neighborhood U_M such that the restriction of π_S to U_M is a trivial principal bundle with the structure additive group (1.1), and we have a composite fibre bundle

$$\widehat{S} : U_M \longrightarrow \pi_S(U_M) \longrightarrow \widehat{S}(U_M) \subset \mathbb{R}^m. \quad (2.10)$$

- (III) A neighborhood U_M is provided with the bundle (generalized action-angle) coordinates

$$(I_\lambda, q^A, y^\lambda) \rightarrow (I_\lambda, q^A) \rightarrow (I_\lambda), \quad \lambda = 1, \dots, m, \quad A = 1, \dots, 2(n-m), \quad (2.11)$$

such that: (i) the action coordinates (I_λ) on $\widehat{S}(U_M)$ are expressed into the values of generating functions S_λ ; (ii) the angle coordinates (y^λ) are coordinates on the toroidal cylinder (1.1); and (iii) a symplectic form Ω on U_M reads

$$\Omega = dI_\lambda \wedge dy^\lambda + \Omega_{AB}(I_\beta, q^C) dq^A \wedge dq^B.$$

□

3 Lie algebra superintegrable systems

Following the original Mishchenko–Fomenko theorem [3, 4, 20], let us consider superintegrable systems whose generating functions $F = \{F_1, \dots, F_k\}$ form a k -dimensional real Lie algebra \mathfrak{g} of corank $m = 2n - k$ with the commutation relations

$$\{F_i, F_j\} = c_{ij}^h F_h, \quad c_{ij}^h = \text{const.} \quad (3.1)$$

We agree to call them the Lie algebra superintegrable systems. In this case, the fibration \widehat{F} (2.2) is a momentum mapping of Z onto a domain N of the Lie coalgebra \mathfrak{g}^* (Section 6.2) which is provided with the coordinates x_i in item (i) of Definition 2.1 [12, 13, 26]. Accordingly, the coinduced Poisson structure $\{\cdot, \cdot\}_N$ on N coincides with the canonical Lie–Poisson structure on \mathfrak{g}^* , and it is given by a Poisson bivector field

$$w = \frac{1}{2} c_{ij}^h x_h \partial^i \wedge \partial^j.$$

In view of the relations (6.19), Hamiltonian vector fields ϑ_i of generating functions F_i of the Lie algebra superintegrable system (3.1) make up a real Lie algebra \mathfrak{g} with the commutation relations

$$[\vartheta_i, \vartheta_j] = c_{ij}^h \vartheta_h. \quad (3.2)$$

Since the morphism w^\sharp (6.16) is of maximal rank, Hamiltonian vector fields ϑ_i are independent, i.e., $\bigwedge^k \vartheta_i \neq 0$.

Following the conditions of Theorem 2.3, let us assume that Hamiltonian vector fields ϑ_i are complete. In accordance with the above mentioned theorem [22, 23], they define a Hamiltonian action on Z of a simply connected Lie group G whose Lie algebra is isomorphic to \mathfrak{g} . Since vector fields ϑ_i are independent, the action of G on Z is locally free, i.e., isotropy groups of points of U are discrete subgroups of G . Orbits of G coincide with k -dimensional maximal integral manifolds of a regular distribution on Z spanned by Hamiltonian vector fields ϑ_i [27]. They constitute a foliation \mathcal{F} of Z . Then the fibration \widehat{F} (2.2) sends its leaves \mathcal{F}_z through points $z \in Z$ onto the orbits $\mathcal{G}_{\widehat{F}(z)}$ of the coadjoint action (6.10) of G on \mathfrak{g}^* , which coincide with the canonical symplectic foliation \mathcal{G} of \mathfrak{g}^* . Conversely, $\mathcal{F}_z = \widehat{F}^{-1}(\mathcal{G}_{\widehat{F}(z)})$ in accordance with item (iii) of Lemma 2.2.

It should be noted that Casimir functions $C \in \mathcal{C}(\mathfrak{g}^*)$ of the Lie–Poisson structure on \mathfrak{g}^* are exactly the coadjoint invariant functions on \mathfrak{g}^* . They are constant on orbits of the coadjoint action of G on \mathfrak{g}^* . Consequently, their pull-back \widehat{F}^*C are constant on leaves of a foliation \mathcal{F} . Therefore, the real Lie algebra \mathfrak{g} (3.2) is extended to a Lie algebra over a subring $\mathcal{C} = \widehat{F}^*\mathcal{C}(N) \subset C^\infty(Z)$ of the pull-back $\widehat{F}C$ of Casimir functions on $N \subset \mathfrak{g}^*$.

Now let us assume that a foliation \mathcal{F} is a fibred manifold $\pi_{\mathcal{F}} : Z \rightarrow \pi_{\mathcal{F}}(Z)$ whose fibres are mutually diffeomorphic. This implies that a symplectic foliation \mathcal{G} of $\widehat{F}(Z)$ also is a fibred manifold $\pi_{\mathcal{G}} : \widehat{F}(Z) \rightarrow \pi_{\mathcal{F}}(Z)$. Thus, we have a

composite fibred manifold

$$\pi_{\mathcal{F}} = \pi_{\mathcal{G}} \circ \widehat{F} : Z \longrightarrow \widehat{F}(Z) \longrightarrow \pi_{\mathcal{F}}(Z), \quad (3.3)$$

so that a fibration \widehat{F} obeys the conditions of Theorem 2.3. It follows that, given a leaf V of \mathcal{F} , there exists its open saturated neighborhood U_V such that $\widehat{F}(U_V) \subset \mathfrak{g}^*$ is provided with some family of m independent Casimir functions $C = (C_1, \dots, C_m)$ and, restricted to U_V , the composite fibred manifold $\pi_{\mathcal{F}}$ (3.3) becomes a composite bundle

$$\begin{aligned} \pi_{\mathcal{F}} = \widehat{C} \circ \widehat{F} : U_V &\longrightarrow \widehat{F}(U_V) \longrightarrow \times^m C_{\lambda}(\widehat{F}(U_V)) = \\ &\times^m \widehat{F}^* C_{\lambda}(U_V) = \pi_{\mathcal{F}}(U_V) \end{aligned} \quad (3.4)$$

in toroidal cylinders. In accordance with Remark 2.2, fibres of \widehat{F} are integral manifolds of a commutative partially integrable system $S = (S_{\lambda})$ on Z of the pull-back $S_{\lambda} = \widehat{F}^* C_{\lambda}$ of Casimir functions C_{λ} . Then the composite bundle (3.4) takes the form (2.10). Accordingly, it is endowed with the bundle (generalized action-angle) coordinates

$$(I_{\lambda}, q^A, y^{\lambda}) \rightarrow (I_{\lambda}, q^A) \rightarrow (I_{\lambda}), \quad \lambda = 1, \dots, m, \quad A = 1, \dots, k - m, \quad (3.5)$$

which are generalized action-angle coordinates (2.11) in Theorem 2.6 when S_{λ} are the pull-back $\widehat{F}^* C_{\lambda}$ of Casimir functions C_{λ} . Note that the latter in turn are the pull-back $C_{\lambda} = \pi_{\mathcal{G}}^* \Phi_{\lambda}$ of some functions Φ_{λ} on a base $\pi_{\mathcal{F}}(U_V)$ of the composite bundle (3.4).

4 Commutative partially integrable systems on Poisson manifolds

As was mentioned above invariant submanifolds of a superintegrable system are integral manifolds of a certain commutative partially integrable system on a symplectic manifold (Remark 2.2). Therefore, we start our analysis of partially superintegrable systems with commutative partially integrable systems on Poisson manifolds in Example 1.3 [5, 10, 16, 12, 26].

A key point is that a commutative partially integrable system admits different compatible Poisson structures (Theorem 4.2). Treating commutative partially integrable systems, we therefore are based on a wider notion of the commutative dynamical algebra [10].

Let us have m mutually commutative vector fields $\{\vartheta_{\lambda}\}$ on a connected smooth manifold Z which are functionally independent (i.e., $\bigwedge^m \vartheta_{\lambda} \neq 0$) everywhere on Z . We denote by $\mathcal{C} \subset C^{\infty}(Z)$ a \mathbb{R} -subring of smooth real functions f on Z whose derivations $\vartheta_{\lambda} \rfloor df$ along ϑ_i vanish for all ϑ_{λ} . Let \mathcal{A} be an m -dimensional Lie \mathcal{C} -algebra generated by the vector fields $\{\vartheta_{\lambda}\}$.

DEFINITION 4.1: We agree to call \mathcal{A} the commutative dynamical algebra. \square

For instance, given a commutative partially integrable system S on a symplectic manifold (Remark 2.1), the Hamiltonian vector fields of its generating functions constitute a commutative dynamical algebra in accordance with Definition 4.1.

In a general setting, let us now consider a commutative dynamical algebra on a Poisson manifold.

DEFINITION 4.2: Let (Z, w) be a (regular) Poisson manifold (Section 6.4) and \mathcal{A} an m -dimensional commutative dynamical algebra on Z . A triple (Z, \mathcal{A}, w) is said to be a commutative partially integrable system if the following hold.

(a) The generators ϑ_λ are Hamiltonian vector fields of some independent functions $S_\lambda \in \mathcal{C}$ on Z .

(b) All elements of $\mathcal{C} \subset C^\infty(Z)$ are mutually in involution, i.e., their Poisson brackets $\{f, f'\}_w$, $f, f' \in \mathcal{C}$, equal zero. \square

It follows at once from this definition that the Poisson structure w is at least of rank $2m$, and that \mathcal{C} is a commutative Poisson algebra. We call the functions S_λ in item (a) of Definition 4.2 the generating functions of a commutative partially integrable system, which is uniquely defined by a family (S_1, \dots, S_m) of these functions.

DEFINITION 4.3: We say that a Poisson structure in Definition 4.1 is compatible. \square

One can show (Theorem 4.2) that a compatible Poisson structure is of rank $2m$.

Remark 4.1: If $2m = \dim Z$ in Definition 4.2, we have a completely integrable system on a symplectic manifold Z (Definition 2.2). However, a commutative partially integrable system on a symplectic manifold in Remark 2.1 fails to be well in accordance with Definition 4.2 because it does not satisfy item (b) of this Definition if $m < n$. \square

If $2m < \dim Z$, there exist different compatible Poisson structures on Z which bring a commutative dynamical algebra \mathcal{A} into a commutative partially integrable system.

Forthcoming Theorem 4.1 shows that, under certain conditions, there exists a compatible Poisson structure on an open neighborhood of an invariant submanifold M of a commutative dynamical algebra. Theorems 4.2 – 4.3 describe all these Poisson structures around an invariant submanifold $M \subset Z$ of \mathcal{A} [10]. Given a commutative partially integrable system (w, \mathcal{A}) in Theorem 4.2, the bivector field w (4.9) can be brought into the canonical form (4.7) with respect to generalized action-angle coordinates in Theorem 4.5. This theorem extends the above-mentioned Liouville–Arnold and Poincaré–Lyapounov–Nekhoroshev theorems to the case of a Poisson structure and a non-compact invariant submanifold [10, 12, 26].

Given a commutative dynamical algebra \mathcal{A} on a manifold Z , let G be the group of local diffeomorphisms of Z generated by the flows of its elements. The orbits of G are maximal invariant submanifolds of \mathcal{A} (we follow the terminology

of [27]). Tangent spaces to these submanifolds form a (regular) distribution $\mathcal{V} \subset TZ$ whose maximal integral manifolds coincide with orbits of G . Being involutive, this distribution yields a foliation \mathcal{S} of Z (Section 6.1).

THEOREM 4.1: Let \mathcal{A} be a commutative dynamical algebra, M its invariant submanifold, and U a saturated open neighborhood of M (Remark 2.3). Let us suppose that:

- (i) vector fields ϑ_λ on U are complete,
- (ii) a foliation \mathcal{S} of U is a fibred manifold $\pi_{\mathcal{S}}$ with mutually diffeomorphic fibres.

Then the following hold [10, 12, 26].

- (I) Leaves of \mathcal{S} are diffeomorphic to the toroidal cylinder (1.1).
- (II) There exists an open saturated neighborhood of M , say U again, which is a trivial principal bundle

$$U = N \times (\mathbb{R}^{m-r} \times T^r) \xrightarrow{\pi_{\mathcal{S}}} N \quad (4.1)$$

with the structure additive group (1.1) over a domain $N \subset \mathbb{R}^{\dim Z - m}$.

- (III) If $2m \leq \dim Z$, there exists a Poisson structure of rank $2m$ on U such that \mathcal{A} is a commutative partially integrable system in accordance with Definition 4.2. \square

Outline of proof: (I) Since m -dimensional leaves of the foliation \mathcal{F} admit m complete independent vector fields, they are locally affine manifolds diffeomorphic to the toroidal cylinder (1.1).

(II) Since a foliation \mathcal{F} of U is a fibred manifold by virtue of item (ii), one can always choose an open fibred neighborhood of its fibre M , say U again, over a domain N such that this fibred manifold

$$\pi : U \rightarrow N \quad (4.2)$$

admits a section σ . In accordance with the above mentioned theorem [22, 23], complete Hamiltonian vector fields ϑ_λ define an action of a simply connected Lie group G on Z . Because vector fields ϑ_λ are mutually commutative, it is an additive group \mathbb{R}^m whose group space is coordinated by parameters s^λ of the flows with respect to the basis $\{e_\lambda = \vartheta_\lambda\}$ for its Lie algebra. Orbits of a group \mathbb{R}^m in $U \subset Z$ coincide with fibres of the fibred manifold (4.2). Since vector fields ϑ_λ are independent everywhere on U , the action of \mathbb{R}^m on U is locally free, i.e., isotropy groups of points of U are discrete subgroups of a group \mathbb{R}^m . Given a point $x \in N$, the action of \mathbb{R}^m on a fibre $M_x = \pi^{-1}(x)$ factorizes as

$$\mathbb{R}^m \times M_x \rightarrow G_x \times M_x \rightarrow M_x \quad (4.3)$$

through the free transitive action on M_x of the factor group $G_x = \mathbb{R}^m / K_x$, where K_x is the isotropy group of an arbitrary point of M_x . It is the same group for all points of M_x because \mathbb{R}^m is a commutative group. Clearly, M_x is diffeomorphic to a group space of G_x . Since fibres M_x are mutually diffeomorphic, all isotropy groups K_x are isomorphic to the group \mathbb{Z}_r for some fixed

$0 \leq r \leq m$. Accordingly, the groups G_x are isomorphic to the additive group (1.1). Let us bring the fibred manifold (4.2) into a principal bundle with a structure group G_0 , where we denote $\{0\} = \pi(M)$. For this purpose, let us determine isomorphisms $\rho_x : G_0 \rightarrow G_x$ of a group G_0 to groups G_x , $x \in N$. Then a desired fibrewise action of G_0 on U is defined by the law

$$G_0 \times M_x \rightarrow \rho_x(G_0) \times M_x \rightarrow M_x. \quad (4.4)$$

Generators of each isotropy subgroup K_x of \mathbb{R}^m are given by r linearly independent vectors of the group space \mathbb{R}^m . One can show that there exist ordered collections of generators $(v_1(x), \dots, v_r(x))$ of the groups K_x such that $x \rightarrow v_i(x)$ are smooth \mathbb{R}^m -valued fields on N . Indeed, given a vector $v_i(0)$ and a section σ of the fibred manifold (4.2), each field $v_i(x) = (s_i^\alpha(x))$ is a unique smooth solution of an equation

$$g(s_i^\alpha)\sigma(x) = \sigma(x), \quad (s_i^\alpha(0)) = v_i(0),$$

on an open neighborhood of $\{0\}$. Let us consider the decomposition

$$v_i(0) = B_i^a(0)e_a + C_i^j(0)e_j, \quad a = 1, \dots, m-r, \quad j = 1, \dots, r,$$

where $C_i^j(0)$ is a non-degenerate matrix. Since the fields $v_i(x)$ are smooth, there exists an open neighborhood of $\{0\}$, say N again, where the matrices $C_i^j(x)$ are non-degenerate. Then

$$A(x) = \begin{pmatrix} \text{Id} & (B(x) - B(0))C^{-1}(0) \\ 0 & C(x)C^{-1}(0) \end{pmatrix} \quad (4.5)$$

is a unique linear endomorphism

$$(e_a, e_i) \rightarrow (e_a, e_j)A(x)$$

of a vector space \mathbb{R}^m which transforms a frame $\{v_\lambda(0)\} = \{e_a, v_i(0)\}$ into a frame $\{v_\lambda(x)\} = \{e_a, v_i(x)\}$, i.e.,

$$v_i(x) = B_i^a(x)e_a + C_i^j(x)e_j = B_i^a(0)e_a + C_i^j(0)[A_j^b(x)e_b + A_j^k(x)e_k].$$

Since $A(x)$ (4.5) also is an automorphism of a group \mathbb{R}^m sending K_0 onto K_x , we obtain a desired isomorphism ρ_x of a group G_0 to a group G_x . Let an element g of a group G_0 be the coset of an element $g(s^\lambda)$ of a group \mathbb{R}^m . Then it acts on M_x by the rule (4.4) just as the element $g((A_x^{-1})^\lambda_\beta s^\beta)$ of a group \mathbb{R}^m does. Since entries of the matrix A (4.5) are smooth functions on N , this action of a group G_0 on U is smooth. It is free, and $U/G_0 = N$. Then the fibred manifold (4.2) is a trivial principal bundle with a structure group G_0 . Given a section σ of this principal bundle, its trivialization $U = N \times G_0$ is defined by assigning the points $\rho^{-1}(g_x)$ of a group space G_0 to the points $g_x\sigma(x)$, $g_x \in G_x$, of a fibre M_x . Let us endow G_0 with the standard coordinate atlas $(r^\lambda) = (t^a, \varphi^i)$ of the group (1.1). Then U admits the trivialization (4.1) with respect to the bundle

coordinates (x^A, t^a, φ^i) where x^A , $A = 1, \dots, \dim Z - m$, are coordinates on a base N . The vector fields ϑ_λ on U relative to these coordinates read

$$\vartheta_a = \partial_a, \quad \vartheta_i = -(BC^{-1})_i^a(x)\partial_a + (C^{-1})_i^k(x)\partial_k. \quad (4.6)$$

Accordingly, the subring \mathcal{C} restricted to U is the pull-back $\pi^*C^\infty(N)$ onto U of a ring of smooth functions on N .

(III). Let us split coordinates (x^A) on N into some m coordinates (J_λ) and the rest $\dim Z - 2m$ coordinates (z^A) . Then we can provide the toroidal domain U (4.1) with the Poisson bivector field

$$w = \partial^\lambda \wedge \partial_\lambda \quad (4.7)$$

of rank $2m$. The independent complete vector fields ∂_a and ∂_i are Hamiltonian vector fields of the functions $S_a = J_a$ and $S_i = J_i$ on U which are in involution with respect to a Poisson bracket $\{\cdot, \cdot\}_w$ defined by the bivector field w (4.7). By virtue of the expression (4.6), the Hamiltonian vector fields $\{\partial_\lambda\}$ generate the \mathcal{C} -algebra \mathcal{A} . Therefore, (w, \mathcal{A}) is a commutative partially integrable system on a Poisson manifold (Z, w) . \square

Remark 4.2: If fibres of a fibred manifold in item (ii) of Theorem 4.1 are assumed to be compact then this fibred manifold is a fibre bundle and vertical vector fields on it (e.g., in condition (i) of Theorem 4.1) are complete. \square

A Poisson structure in Theorem 4.1 is by no means unique as follows.

THEOREM 4.2: Given the toroidal domain U (4.1) provided with bundle coordinates (x^A, r^λ) , it is readily observed that, if a Poisson bivector field on U satisfies Definition 4.2, it takes a form

$$w = w_1 + w_2 = w^{A\lambda}(x^B)\partial_A \wedge \partial_\lambda + w^{\mu\nu}(x^B, r^\lambda)\partial_\mu \wedge \partial_\nu. \quad (4.8)$$

Conversely, given a Poisson bivector field w (4.8) of rank $2m$ on the toroidal domain U (4.1), there exists a toroidal domain $U' \subset U$ such that a commutative dynamical algebra \mathcal{A} in Theorem 4.1 is a commutative partially integrable system on U' . \square

Remark 4.3: It is readily observed that any Poisson bivector field w (4.8) fulfils condition (b) in Definition 4.2, but condition (a) imposes a restriction on a toroidal domain U . A key point is that the characteristic foliation \mathcal{F} of U yielded by the Poisson bivector fields w (4.8) is the pull-back of an m -dimensional foliation \mathcal{F}_N of a base N , which is defined by the first summand w_1 (4.8) of w . With respect to the adapted coordinates (J_λ, z^A) , $\lambda = 1, \dots, m$, on the foliated manifold (N, \mathcal{F}_N) , the Poisson bivector field w reads

$$w = w_\nu^\mu(J_\lambda, z^A)\partial^\nu \wedge \partial_\mu + w^{\mu\nu}(J_\lambda, z^A, r^\lambda)\partial_\mu \wedge \partial_\nu. \quad (4.9)$$

Then condition (a) in Definition 4.2 is satisfied if $N' \subset N$ is a domain of a coordinate chart (J_λ, z^A) of the foliation \mathcal{F}_N . In this case, a commutative

dynamical algebra \mathcal{A} on a toroidal domain $U' = \pi^{-1}(N')$ is generated by the Hamiltonian vector fields

$$\vartheta_\lambda = -w[dJ_\lambda = w^\mu_\lambda \partial_\mu] \quad (4.10)$$

of the m independent functions $S_\lambda = J_\lambda$. \square

Outline of proof: The characteristic distribution of the Poisson bivector field w (4.8) is spanned by Hamiltonian vector fields

$$v^A = -w[dx^A = w^{A\mu} \partial_\mu] \quad (4.11)$$

and vector fields

$$w[dr^\lambda = w^{A\lambda} \partial_A + 2w^{\mu\lambda} \partial_\mu].$$

Since w is of rank $2m$, the vector fields ∂_μ can be expressed in the vector fields v^A (4.11). Hence, the characteristic distribution of w is spanned by the Hamiltonian vector fields v^A (4.11) and the vector fields

$$v^\lambda = w^{A\lambda} \partial_A. \quad (4.12)$$

The vector fields (4.12) are projected onto N . Moreover, one can derive from the relation $[w, w]_{\text{SN}} = 0$ that they generate a Lie algebra and, consequently, span an involutive distribution \mathcal{V}_N of rank m on N . Let \mathcal{F}_N denote the corresponding foliation of N . We consider the pull-back $\mathcal{F} = \pi^* \mathcal{F}_N$ of this foliation onto U by the trivial fibration π . Its leaves are the inverse images $\pi^{-1}(F_N)$ of leaves F_N of the foliation \mathcal{F}_N , and so is its characteristic distribution

$$T\mathcal{F} = (T\pi)^{-1}(\mathcal{V}_N).$$

This distribution is spanned by the vector fields v^λ (4.12) on U and the vertical vector fields on $U \rightarrow N$, namely, the vector fields v^A (4.11) generating a commutative dynamical algebra \mathcal{A} . Hence, $T\mathcal{F}$ is the characteristic distribution of a Poisson bivector field w . Furthermore, since $U \rightarrow N$ is a trivial bundle, each leaf $\pi^{-1}(F_N)$ of the pull-back foliation \mathcal{F} is the manifold product of a leaf F_N of N and the toroidal cylinder (1.1). It follows that the foliated manifold (U, \mathcal{F}) can be provided with an adapted coordinate atlas

$$\{(U_\iota, J_\lambda, z^A, r^\lambda)\}, \quad \lambda = 1, \dots, m, \quad A = 1, \dots, \dim Z - 2m,$$

such that (J_λ, z^A) are adapted coordinates on the foliated manifold (N, \mathcal{F}_N) . Relative to these coordinates, the Poisson bivector field (4.8) takes the form (4.9). Let N' be the domain of this coordinate chart. Then a commutative dynamical algebra \mathcal{A} on a toroidal domain $U' = \pi^{-1}(N')$ is generated by the Hamiltonian vector fields ϑ_λ (4.10) of functions $S_\lambda = J_\lambda$. \square

Remark 4.4: Let us note that coefficients $w^{\mu\nu}$ in the expressions (4.8) and (4.9) are affine in coordinates r^λ because of the relation $[w, w]_{\text{SN}} = 0$ and, consequently, they are constant on tori. \square

Now, let w and w' be two different Poisson structures (4.8) on the toroidal domain (4.1) which make a commutative dynamical algebra \mathcal{A} into different commutative partially integrable systems (w, \mathcal{A}) and (w', \mathcal{A}) .

DEFINITION 4.4: We agree to call a triple (w, w', \mathcal{A}) the bi-Poisson commutative partially integrable system if any Hamiltonian vector field $\vartheta \in \mathcal{A}$ with respect to w possesses the same Hamiltonian representation

$$\vartheta = -w \lrcorner df = -w' \lrcorner df, \quad f \in \mathcal{C}, \quad (4.13)$$

relative to w' , and *vice versa*. \square

Definition 4.4 establishes *sui generis* an equivalence between the commutative partially integrable systems (w, \mathcal{A}) and (w', \mathcal{A}) .

THEOREM 4.3: (I) The triple (w, w', \mathcal{A}) is a bi-Poisson partially integrable system in accordance with Definition 4.4 iff the Poisson bivector fields w and w' (4.8) differ in the second terms w_2 and w'_2 . (II) These Poisson bivector fields admit a recursion operator. \square

Outline of proof: (I). It is easily justified that, if Poisson bivector fields w (4.8) fulfil Definition 4.4, they are distinguished only by the second summand w_2 . Conversely, as follows from the proof of Theorem 4.2, the characteristic distribution of the Poisson bivector field w (4.8) is spanned by the vector fields (4.11) and (4.12). Hence, all Poisson bivector fields w (4.8) distinguished only by the second summand w_2 have the same characteristic distribution, and they bring \mathcal{A} into a commutative partially integrable system on the same toroidal domain U' . Then the condition in Definition 4.4 is easily justified. (II). The result follows from forthcoming Lemma 4.4. \square

Given a smooth real manifold X , let w and w' be Poisson bivector fields of rank $2m$ on X , and let w^\sharp and w'^\sharp be the corresponding bundle homomorphisms (6.16). A tangent-valued one-form R on X yields bundle endomorphisms

$$R : TX \rightarrow TX, \quad R^* : T^*X \rightarrow T^*X. \quad (4.14)$$

It is called the recursion operator if

$$w'^\sharp = R \circ w^\sharp = w^\sharp \circ R^*. \quad (4.15)$$

LEMMA 4.4: A recursion operator between Poisson structures of the same rank exists iff their characteristic distributions coincide. \square

Outline of proof: It follows from the equalities (4.15) that a recursion operator R sends the characteristic distribution of w to that of w' , and these distributions coincide if w and w' are of the same rank. Conversely, let Poisson structures w and w' possess the same characteristic distribution $T\mathcal{F} \rightarrow TX$ tangent to a foliation \mathcal{F} of X . We have the exact sequences (6.2) – (6.3). The bundle homomorphisms w^\sharp and w'^\sharp (6.16) factorize in the unique fashion (6.23) through the bundle isomorphisms $w_{\mathcal{F}}^\sharp$ and $w'_{\mathcal{F}}^\sharp$ (6.23). Let us consider inverse isomorphisms

$$w_{\mathcal{F}}^\flat : T\mathcal{F} \rightarrow T\mathcal{F}^*, \quad w'_{\mathcal{F}}^\flat : T\mathcal{F} \rightarrow T\mathcal{F}^* \quad (4.16)$$

and compositions

$$R_{\mathcal{F}} = w'_{\mathcal{F}} \circ w_{\mathcal{F}}^b : T\mathcal{F} \rightarrow T\mathcal{F}, \quad R_{\mathcal{F}}^* = w_{\mathcal{F}}^b \circ w'_{\mathcal{F}} : T\mathcal{F}^* \rightarrow T\mathcal{F}^*. \quad (4.17)$$

There is an obvious relation

$$w'_{\mathcal{F}} = R_{\mathcal{F}} \circ w_{\mathcal{F}}^{\sharp} = w_{\mathcal{F}}^{\sharp} \circ R_{\mathcal{F}}^*.$$

In order to obtain a recursion operator (4.15), it suffices to extend the morphisms $R_{\mathcal{F}}$ and $R_{\mathcal{F}}^*$ (4.17) onto TX and T^*X , respectively. For this purpose, let us consider a splitting

$$\zeta : TX \rightarrow T\mathcal{F}, \quad TX = T\mathcal{F} \oplus (\text{Id} - i_{\mathcal{F}} \circ \zeta)TX = T\mathcal{F} \oplus E,$$

of the exact sequence (6.2) and the dual splitting

$$\zeta^* : T\mathcal{F}^* \rightarrow T^*X, \quad T^*X = \zeta^*(T\mathcal{F}^*) \oplus (\text{Id} - \zeta^* \circ i_{\mathcal{F}}^*)T^*X = \zeta^*(T\mathcal{F}^*) \oplus E',$$

of the exact sequence (6.3). Then the desired extensions are

$$R = R_{\mathcal{F}} \times \text{Id } E, \quad R^* = (\zeta^* \circ R_{\mathcal{F}}^*) \times \text{Id } E'.$$

This recursion operator is invertible, i.e., the morphisms (4.14) are bundle isomorphisms. \square

For instance, the Poisson bivector field w (4.8) and the Poisson bivector field

$$w_0 = w^{A\lambda} \partial_A \wedge \partial_{\lambda}$$

admit a recursion operator $w_0^{\sharp} = R \circ w^{\sharp}$ whose entries are given by the equalities

$$R_B^A = \delta_B^A, \quad R_{\nu}^{\mu} = \delta_{\nu}^{\mu}, \quad R_{\lambda}^A = 0, \quad w^{\mu\lambda} = R_B^{\lambda} w^{B\mu}.$$

Given a commutative partially integrable system (w, \mathcal{A}) in Theorem 4.2, the bivector field w (4.9) can be brought into the canonical form (4.7) with respect to partial action-angle coordinates in forthcoming Theorem 4.5. This theorem extends the Liouville–Arnold theorem to the case of a Poisson structure and a non-compact invariant submanifold [10, 12, 26].

THEOREM 4.5: Given a commutative partially integrable system (w, \mathcal{A}) on a Poisson manifold (U, w) , there exists a toroidal domain $U' \subset U$ equipped with partial action-angle coordinates $(I_a, I_i, z^A, \tau^a, \phi^i)$ such that, restricted to U' , a Poisson bivector field takes the canonical form

$$w = \partial^a \wedge \partial_a + \partial^i \wedge \partial_i, \quad (4.18)$$

while a commutative dynamical algebra \mathcal{A} is generated by Hamiltonian vector fields of the action coordinate functions $S_a = I_a$, $S_i = I_i$. \square

Outline of proof: First, let us employ Theorem 4.2 and restrict U to a toroidal domain, say U again, equipped with coordinates $(J_{\lambda}, z^A, r^{\lambda})$ such that a Poisson

bivector field w takes the form (4.9) and a commutative dynamical algebra \mathcal{A} is generated by the Hamiltonian vector fields ϑ_λ (4.10) of m independent functions $S_\lambda = J_\lambda$ in involution. Let us choose these vector fields as new generators of a group G and return to Theorem 4.1. In accordance with this theorem, there exists a toroidal domain $U' \subset U$ provided with another trivialization $U' \rightarrow N' \subset N$ in the toroidal cylinders (1.1) and endowed with bundle coordinates $(J_\lambda, z^A, r^\lambda)$ such that the vector fields ϑ_λ (4.10) take the form (4.6). For the sake of simplicity, let U' , N' and y^λ be denoted U , N and $r^\lambda = (t^a, \varphi^i)$ again. Herewith, a Poisson bivector field w is given by the expression (4.9) with new coefficients. Let $w^\sharp : T^*U \rightarrow TU$ be the corresponding bundle homomorphism. It factorizes in a unique fashion (6.23):

$$w^\sharp : T^*U \xrightarrow{i_{\mathcal{F}}^*} T\mathcal{F}^* \xrightarrow{w_{\mathcal{F}}^\sharp} T\mathcal{F} \xrightarrow{i_{\mathcal{F}}} TU$$

through the bundle isomorphism

$$w_{\mathcal{F}}^\sharp : T\mathcal{F}^* \rightarrow T\mathcal{F}, \quad w_{\mathcal{F}}^\sharp : \alpha \rightarrow -w(x)|\alpha.$$

Then the inverse isomorphisms $w_{\mathcal{F}}^\flat : T\mathcal{F} \rightarrow T\mathcal{F}^*$ provides a foliated manifold (U, \mathcal{F}) with the leafwise symplectic form

$$\Omega_{\mathcal{F}} = \Omega^{\mu\nu}(J_\lambda, z^A, t^a) \tilde{d}J_\mu \wedge \tilde{d}J_\nu + \Omega_\mu^\nu(J_\lambda, z^A) \tilde{d}J_\nu \wedge \tilde{d}r^\mu, \quad (4.19)$$

$$\Omega_\mu^\alpha w_\beta^\mu = \delta_\beta^\alpha, \quad \Omega^{\alpha\beta} = -\Omega_\mu^\alpha \Omega_\nu^\beta w^{\mu\nu}. \quad (4.20)$$

Let us show that it is \tilde{d} -exact. Let F be a leaf of the foliation \mathcal{F} of U . There is a homomorphism of the de Rham cohomology $H_{\text{DR}}^*(U)$ of U to the de Rham cohomology $H_{\text{DR}}^*(F)$ of F , and it factorizes through the leafwise cohomology $H_{\mathcal{F}}^*(U)$. Since N is a domain of an adapted coordinate chart of the foliation \mathcal{F}_N , the foliation \mathcal{F}_N of N is a trivial fibre bundle

$$N = V \times W \rightarrow W.$$

Since \mathcal{F} is the pull-back onto U of the foliation \mathcal{F}_N of N , it also is a trivial fibre bundle

$$U = V \times W \times (\mathbb{R}^{k-m} \times T^m) \rightarrow W \quad (4.21)$$

over a domain $W \subset \mathbb{R}^{\dim Z - 2m}$. It follows that

$$H_{\text{DR}}^*(U) = H_{\text{DR}}^*(T^r) = H_{\mathcal{F}}^*(U).$$

Then the closed leafwise two-form $\Omega_{\mathcal{F}}$ (4.19) is exact due to the absence of the term $\Omega_{\mu\nu} dr^\mu \wedge dr^\nu$. Moreover, $\Omega_{\mathcal{F}} = \tilde{d}\Xi$ where Ξ reads

$$\Xi = \Xi^\alpha(J_\lambda, z^A, r^\lambda) \tilde{d}J_\alpha + \Xi_i(J_\lambda, z^A) \tilde{d}\varphi^i$$

up to a \tilde{d} -exact leafwise form. The Hamiltonian vector fields $\vartheta_\lambda = \vartheta_\lambda^\mu \partial_\mu$ (4.6) obey the relation

$$\vartheta_\lambda \rfloor \Omega_{\mathcal{F}} = -\tilde{d}J_\lambda, \quad \Omega_\beta^\alpha \vartheta_\lambda^\beta = \delta_\lambda^\alpha, \quad (4.22)$$

which falls into the following conditions

$$\Omega_i^\lambda = \partial^\lambda \Xi_i - \partial_i \Xi^\lambda, \quad (4.23)$$

$$\Omega_a^\lambda = -\partial_a \Xi^\lambda = \delta_a^\lambda. \quad (4.24)$$

The first of the relations (4.20) shows that Ω_β^α is a non-degenerate matrix independent of coordinates r^λ . Then the condition (4.23) implies that $\partial_i \Xi^\lambda$ are independent of φ^i , and so are Ξ^λ since φ^i are cyclic coordinates. Hence,

$$\Omega_i^\lambda = \partial^\lambda \Xi_i, \quad (4.25)$$

$$\partial_i \rfloor \Omega_{\mathcal{F}} = -\tilde{d}\Xi_i. \quad (4.26)$$

Let us introduce new coordinates $I_a = J_a$, $I_i = \Xi_i(J_\lambda)$. By virtue of the equalities (4.24) and (4.25), the Jacobian of this coordinate transformation is regular. The relation (4.26) shows that ∂_i are Hamiltonian vector fields of the functions $S_i = I_i$. Consequently, we can choose vector fields ∂_λ as generators of a commutative dynamical algebra \mathcal{A} . One obtains from the equality (4.24) that

$$\Xi^a = -t^a + E^a(J_\lambda, z^A)$$

and Ξ^i are independent of t^a . Then the leafwise Liouville form Ξ reads

$$\Xi = (-t^a + E^a(I_\lambda, z^A))\tilde{d}I_a + E^i(I_\lambda, z^A)\tilde{d}I_i + I_i\tilde{d}\varphi^i.$$

The coordinate shifts

$$\tau^a = -t^a + E^a(I_\lambda, z^A), \quad \phi^i = \varphi^i - E^i(I_\lambda, z^A)$$

bring the leafwise form $\Omega_{\mathcal{F}}$ (4.19) into the canonical form

$$\Omega_{\mathcal{F}} = \tilde{d}I_a \wedge \tilde{d}\tau^a + \tilde{d}I_i \wedge \tilde{d}\phi^i$$

which ensures the canonical form (4.18) of a Poisson bivector field w . \square

5 Partially superintegrable systems on Poisson manifolds

Studying partially superintegrable systems, we bear in mind that in Example 1.2, but restrict our consideration to Lie algebra superintegrable systems whose generating functions constitute a real Lie algebra (Section 3).

Given a smooth manifold Z , let $\{\vartheta_i\}$ be k independent vector fields on Z (i.e., $\wedge^k \vartheta_i \neq 0$) which generate a real Lie algebra \mathfrak{g} with commutation relations

$$[\vartheta_i, \vartheta_j] = c_{ij}^h \vartheta_h. \quad (5.1)$$

We denote by $\mathcal{C} \subset C^\infty(Z)$ a \mathbb{R} -subring of smooth real functions f on Z whose derivations $\vartheta_i \rfloor df$ vanish for all ϑ_i . Let \mathcal{A} be a k -dimensional Lie \mathcal{C} -algebra generated by vector fields $\{\vartheta_i\}$.

DEFINITION 5.1: We agree to call \mathcal{A} the dynamical algebra. \square

In particular, this definition reproduces Definition 4.1 of a commutative dynamical algebra if vector fields $\{\vartheta_i\}$ mutually commute.

DEFINITION 5.2: Let (Z, w) be a Poisson manifold and \mathcal{A} a k -dimensional dynamical algebra on Z . A triple (Z, \mathcal{A}, w) is said to be a partially superintegrable system if the following hold.

(a) Generators ϑ_i of \mathcal{A} are Hamiltonian vector fields of some independent functions F_i on Z . In view of the relations (6.19), these functions obey the commutation relations

$$\{F_i, F_j\}_w = c_{ij}^h F_h, \quad \{F_i, f\}_w = 0, \quad f \in \mathcal{C}. \quad (5.2)$$

(b) All elements of $\mathcal{C} \subset C^\infty(Z)$ are mutually in involution, i.e., their Poisson brackets $\{f, f'\}_w$ equal zero. \square

For instance, let $F = (F_i)$ be the Lie algebra superintegrable system (3.1) on a symplectic manifold (Z, Ω) . Hamiltonian vector fields ϑ_i of its generating functions F_i obey the commutation relations (5.1) and yield a dynamical algebra. Then it is a partially superintegrable system in accordance with Definition 5.2 where a ring \mathcal{C} consists of the pull-back of Casimir functions of the Lie–Poisson structure on a Lie coalgebra \mathfrak{g}^* .

A partially superintegrable systems on a product of manifolds in Examples 1.2 and commutative partially integrable systems in Example 1.3 also are well in accordance with Definition 5.2.

Certainly, a Poisson structure w in Definition 5.2 is not unique (see Theorem 4.2 for a case of commutative partially integrable systems). Following Definition 4.3, we agree to call it compatible. Generalizing Theorem 4.3, one can show that two Poisson structures are compatible only if they admit the recursion operator (4.14). In accordance with Lemma 4.4, their symplectic foliations coincide.

LEMMA 5.1: The rank of a compatible Poisson structure of a partially superintegrable system equals $k + m$ where m is a corank of the Lie algebra \mathfrak{g} (5.1). \square

Outline of proof: Let \mathcal{W} be a symplectic foliation of Z and W its leaf. The generating vector field ϑ_i obviously are tangent to W . Restricted to W , they generate a Lie algebra superintegrable system so that $k + m$ is a dimension of a leaf W . \square

Let (Z, \mathcal{A}, w) be a partially superintegrable system, and let its generating vector fields be complete. In accordance with the above-mentioned theorem [22, 23], they define a Hamiltonian action of a simply connected Lie group G whose Lie algebra is isomorphic to \mathfrak{g} on Z . Since vector fields ϑ_i are independent, the action of G on Z is locally free, i.e., isotropy groups of points of U are

discrete subgroups of G . Orbits of G coincide with k -dimensional maximal integral manifolds of a regular distribution \mathcal{V} on Z spanned by vector fields ϑ_i [27]. They constitute a foliation \mathcal{F} of Z . It is subordinate a symplectic foliation \mathcal{W} whose leaves are foliated by leaves of \mathcal{F} .

Let both a foliation \mathcal{F} and a foliation \mathcal{W} be fibred manifolds $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{W}}$ with mutually diffeomorphic fibres, respectively. Thus, we have a composite fibred manifold

$$\pi_{\mathcal{W}} : Z \longrightarrow \pi_{\mathcal{F}}(Z) \longrightarrow \pi_{\mathcal{W}}(Z). \quad (5.3)$$

Then one can show the following.

THEOREM 5.2: Let V be a leaf of \mathcal{F} . Then there exists an open saturated neighborhood U_V of V such that the composite fibred manifold $\pi_{\mathcal{W}}$ (5.3) becomes a composite bundle

$$\pi_{\mathcal{W}} : U_V \longrightarrow \pi_{\mathcal{F}}(U_V) \longrightarrow \pi_{\mathcal{W}}(U_V) \quad (5.4)$$

which is endowed with bundle (generalized action-angle) coordinates

$$(I_{\lambda}, q^A, y^{\lambda}, w^a) \rightarrow (I_{\lambda}, q^A, x^a) \rightarrow (I_{\lambda}, x^a), \quad \lambda = 1, \dots, m, \quad A = 1, \dots, k - m,$$

where: (i) the angle coordinates y^i are coordinates on $\mathbb{R}^{m-r} \times T^r$; (ii) the (I_{λ}, x^a) are coordinates on $\pi_{\mathcal{F}}(U_V)$; the (x^a) are coordinates on $\pi_{\mathcal{W}}(U_V)$. With respect to these coordinates, the Poisson bivector field takes a form

$$w = \partial^{\lambda} \wedge \partial_{\lambda} + w^{A\lambda}(q^B, x^a) \partial_A \wedge \partial_B.$$

□

Outline of proof: The proof of Theorem 5.2 is reduced to Theorems 4.2 and 4.5 for commutative partially integrable systems because the pull-back $\pi_{\mathcal{F}}^*C$ of functions on $\pi_{\mathcal{F}}(U_N)$ constitute a commutative partially integrable system on a Poisson manifold (Z, w) whose integral manifolds, diffeomorphic to toroidal cylinders, are invariant submanifolds of fibres of $\pi_{\mathcal{F}}$. □

6 Appendix

For the convenience of the reader this Section summarize the relevant mathematical material on symplectic manifolds, Poisson manifolds and symplectic foliations [1, 11, 12, 19, 29].

6.1 Distributions and foliations

A subbundle \mathbf{T} of the tangent bundle TZ of a manifold Z is called a regular distribution (or, simply, a distribution). A vector field u on Z is said to be subordinate to a distribution \mathbf{T} if it lives in \mathbf{T} . A distribution \mathbf{T} is called involutive if the Lie bracket of \mathbf{T} -subordinate vector fields also is subordinate to \mathbf{T} .

A subbundle of the cotangent bundle T^*Z of Z is called a codistribution \mathbf{T}^* on a manifold Z . For instance, the annihilator $\text{Ann } \mathbf{T}$ of a distribution \mathbf{T} is a codistribution whose fibre over $z \in Z$ consists of covectors $w \in T_z^*$ such that $v \lrcorner w = 0$ for all $v \in \mathbf{T}_z$.

The following local coordinates can be associated to an involutive distribution [31].

THEOREM 6.1: Let \mathbf{T} be an involutive r -dimensional distribution on a manifold Z , $\dim Z = k$. Every point $z \in Z$ has an open neighborhood U which is a domain of an adapted coordinate chart (z^1, \dots, z^k) such that, restricted to U , the distribution \mathbf{T} and its annihilator $\text{Ann } \mathbf{T}$ are spanned by the local vector fields $\partial/\partial z^1, \dots, \partial/\partial z^r$ and the local one-forms dz^{r+1}, \dots, dz^k , respectively. \square

A connected submanifold N of a manifold Z is called an integral manifold of a distribution \mathbf{T} on Z if $TN \subset \mathbf{T}$. Unless otherwise stated, by an integral manifold is meant an integral manifold of dimension of \mathbf{T} . An integral manifold is called maximal if no different integral manifold contains it. The following is the classical theorem of Frobenius [15, 31].

THEOREM 6.2: Let \mathbf{T} be an involutive distribution on a manifold Z . For any $z \in Z$, there exists a unique maximal integral manifold of \mathbf{T} through z , and any integral manifold through z is its open subset. \square

Maximal integral manifolds of an involutive distribution on a manifold Z are assembled into a regular foliation \mathcal{F} of Z . A regular r -dimensional foliation (or, simply, a foliation) \mathcal{F} of a k -dimensional manifold Z is defined as a partition of Z into connected r -dimensional submanifolds (the leaves of a foliation) F_ι , $\iota \in I$, which possesses the following properties [24, 28].

A manifold Z admits an adapted coordinate atlas

$$\{(U_\xi; z^\lambda, z^i)\}, \quad \lambda = 1, \dots, k-r, \quad i = 1, \dots, r, \quad (6.1)$$

such that transition functions of coordinates z^λ are independent of the remaining coordinates z^i . For each leaf F of a foliation \mathcal{F} , the connected components of $F \cap U_\xi$ are given by the equations $z^\lambda = \text{const}$. These connected components and coordinates (z^i) on them make up a coordinate atlas of a leaf F . It follows that tangent spaces to leaves of a foliation \mathcal{F} constitute an involutive distribution $T\mathcal{F}$ on Z , called the tangent bundle to the foliation \mathcal{F} . The factor bundle $V\mathcal{F} = TZ/T\mathcal{F}$, called the normal bundle to \mathcal{F} , has transition functions independent of coordinates z^i . Let $T\mathcal{F}^* \rightarrow Z$ denote the dual of $T\mathcal{F} \rightarrow Z$. There are the exact sequences

$$0 \rightarrow T\mathcal{F} \xrightarrow{i_{\mathcal{F}}} TX \rightarrow V\mathcal{F} \rightarrow 0, \quad (6.2)$$

$$0 \rightarrow \text{Ann } T\mathcal{F} \rightarrow T^*X \xrightarrow{i_{\mathcal{F}}^*} T\mathcal{F}^* \rightarrow 0 \quad (6.3)$$

of vector bundles over Z .

A pair (Z, \mathcal{F}) , where \mathcal{F} is a foliation of Z , is called a foliated manifold. It should be emphasized that leaves of a foliation need not be closed or imbedded

submanifolds. Every leaf has an open saturated neighborhood U , i.e., if $z \in U$, then a leaf through z also belongs to U .

Any submersion $\zeta : Z \rightarrow M$ yields a foliation

$$\mathcal{F} = \{F_p = \zeta^{-1}(p)\}_{p \in \zeta(Z)}$$

of Z indexed by elements of $\zeta(Z)$, which is an open submanifold of M , i.e., $Z \rightarrow \zeta(Z)$ is a fibred manifold. Leaves of this foliation are closed imbedded submanifolds. Such a foliation is called simple. Any (regular) foliation is locally simple.

Example 6.1: Every smooth real function f on a manifold Z with nowhere vanishing differential df is a submersion $Z \rightarrow \mathbb{R}$. It defines a one-codimensional foliation whose leaves are given by equations

$$f(z) = c, \quad c \in f(Z) \subset \mathbb{R}.$$

This is a foliation of level surfaces of a function f , called the generating function. Every one-codimensional foliation is locally a foliation of level surfaces of some function on Z . The level surfaces of an arbitrary smooth real function f on a manifold Z define a singular foliation \mathcal{F} on Z [14]. Its leaves are not submanifolds in general. Nevertheless if $df(z) \neq 0$, the restriction of \mathcal{F} to some open neighborhood U of z is a foliation with the generating function $f|_U$. \square

Let \mathcal{F} be a (regular) foliation of a k -dimensional manifold Z provided with the adapted coordinate atlas (6.1). The real Lie algebra $\mathcal{T}_1(\mathcal{F})$ of global sections of the tangent bundle $T\mathcal{F} \rightarrow Z$ to \mathcal{F} is a $C^\infty(Z)$ -submodule of the derivation module of the \mathbb{R} -ring $C^\infty(Z)$ of smooth real functions on Z . Its kernel $S_{\mathcal{F}}(Z) \subset C^\infty(Z)$ consists of functions constant on leaves of \mathcal{F} . Therefore, $\mathcal{T}_1(\mathcal{F})$ is the Lie $S_{\mathcal{F}}(Z)$ -algebra of derivations of $C^\infty(Z)$, regarded as a $S_{\mathcal{F}}(Z)$ -ring. Then one can introduce the leafwise differential calculus [9, 11] as the Chevalley–Eilenberg differential calculus over the $S_{\mathcal{F}}(Z)$ -ring $C^\infty(Z)$. It is defined as a subcomplex

$$0 \rightarrow S_{\mathcal{F}}(Z) \rightarrow C^\infty(Z) \xrightarrow{\tilde{d}} \mathfrak{F}^1(Z) \cdots \xrightarrow{\tilde{d}} \mathfrak{F}^{\dim \mathcal{F}}(Z) \rightarrow 0 \quad (6.4)$$

of the Chevalley–Eilenberg complex of the Lie $S_{\mathcal{F}}(Z)$ -algebra $\mathcal{T}_1(\mathcal{F})$ with coefficients in $C^\infty(Z)$ which consists of $C^\infty(Z)$ -multilinear skew-symmetric maps

$$\times^r \mathcal{T}_1(\mathcal{F}) \rightarrow C^\infty(Z), \quad r = 1, \dots, \dim \mathcal{F}.$$

These maps are global sections of exterior products $\overset{r}{\wedge} T\mathcal{F}^*$ of the dual $T\mathcal{F}^* \rightarrow Z$ of $T\mathcal{F} \rightarrow Z$. They are called the leafwise forms on a foliated manifold (Z, \mathcal{F}) , and are given by the coordinate expression

$$\phi = \frac{1}{r!} \phi_{i_1 \dots i_r} \tilde{d}z^{i_1} \wedge \cdots \wedge \tilde{d}z^{i_r},$$

where $\{\tilde{d}z^i\}$ are the duals of the holonomic fibre bases $\{\partial_i\}$ for $T\mathcal{F}$. Then one can think of the Chevalley – Eilenberg coboundary operator

$$\tilde{d}\phi = \tilde{d}z^k \wedge \partial_k \phi = \frac{1}{r!} \partial_k \phi_{i_1 \dots i_r} \tilde{d}z^k \wedge \tilde{d}z^{i_1} \wedge \cdots \wedge \tilde{d}z^{i_r}$$

as being the leafwise exterior differential. Accordingly, the complex (6.4) is called the leafwise de Rham complex (or the tangential de Rham complex).

Let us consider the exact sequence (6.3) of vector bundles over Z . Since it admits a splitting, the epimorphism $i_{\mathcal{F}}^*$ yields that of the algebra $\mathcal{O}^*(Z)$ of exterior forms on Z to the algebra $\mathfrak{F}^*(Z)$ of leafwise forms. It obeys the condition $i_{\mathcal{F}}^* \circ d = \tilde{d} \circ i_{\mathcal{F}}^*$, and provides the cochain morphism

$$\begin{aligned} i_{\mathcal{F}}^* : (\mathbb{R}, \mathcal{O}^*(Z), d) &\rightarrow (S_{\mathcal{F}}(Z), \mathcal{F}^*(Z), \tilde{d}), \\ dz^\lambda &\rightarrow 0, \quad dz^i \rightarrow \tilde{d}z^i, \end{aligned} \quad (6.5)$$

of the de Rham complex of Z to the leafwise de Rham complex (6.4).

Given a leaf $i_F : F \rightarrow Z$ of \mathcal{F} , we have the pull-back homomorphism

$$(\mathbb{R}, \mathcal{O}^*(Z), d) \rightarrow (\mathbb{R}, \mathcal{O}^*(F), d) \quad (6.6)$$

of the de Rham complex of Z to that of F .

PROPOSITION 6.3: The homomorphism (6.6) factorize through the homomorphism [11]. \square

6.2 Differential geometry of Lie groups

Let G be a real Lie group of $\dim G > 0$, and let $L_g : G \rightarrow gG$ and $R_g : G \rightarrow Gg$ denote the action of G on itself by left and right multiplications, respectively. Clearly, L_g and $R_{g'}$ for all $g, g' \in G$ mutually commute, and so do the tangent maps TL_g and $TR_{g'}$.

A vector field ξ_l (resp. ξ_r) on a group G is said to be left-invariant (resp. right-invariant) if $\xi_l \circ L_g = TL_g \circ \xi_l$ (resp. $\xi_r \circ R_g = TR_g \circ \xi_r$). Left-invariant (resp. right-invariant) vector fields make up the left Lie algebra \mathfrak{g}_l (resp. the right Lie algebra \mathfrak{g}_r) of G .

There is one-to-one correspondence between the left-invariant vector field ξ_l (resp. right-invariant vector fields ξ_r) on G and the vectors $\xi_l(e) = TL_{g^{-1}}\xi_l(g)$ (resp. $\xi_r(e) = TR_{g^{-1}}\xi_r(g)$) of the tangent space $T_e G$ to G at the unit element e of G . This correspondence provides $T_e G$ with the left and the right Lie algebra structures. Accordingly, the left action L_g of a Lie group G on itself defines its adjoint representation

$$\xi_r \rightarrow \text{Ad } g(\xi_r) = TL_g \circ \xi_r \circ L_{g^{-1}} \quad (6.7)$$

in the right Lie algebra \mathfrak{g}_r .

Let $\{\epsilon_m\}$ (resp. $\{\varepsilon_m\}$) denote the basis for the left (resp. right) Lie algebra, and let c_{mn}^k be the right structure constants

$$[\varepsilon_m, \varepsilon_n] = c_{mn}^k \varepsilon_k.$$

There is the morphism

$$\rho : \mathfrak{g}_l \ni \epsilon_m \rightarrow \varepsilon_m = -\epsilon_m \in \mathfrak{g}_r$$

between left and right Lie algebras such that

$$[\epsilon_m, \epsilon_n] = -c_{mn}^k \epsilon_k.$$

The tangent bundle $\pi_G : TG \rightarrow G$ of a Lie group G is trivial. There are the following two canonical isomorphisms

$$\begin{aligned} \varrho_l : TG \ni q &\rightarrow (g = \pi_G(q), TL_g^{-1}(q)) \in G \times \mathfrak{g}_l, \\ \varrho_r : TG \ni q &\rightarrow (g = \pi_G(q), TR_g^{-1}(q)) \in G \times \mathfrak{g}_r. \end{aligned}$$

Therefore, any action

$$G \times Z \ni (g, z) \rightarrow gz \in Z$$

of a Lie group G on a manifold Z on the left yields the homomorphism

$$\mathfrak{g}_r \ni \varepsilon \rightarrow \xi_\varepsilon \in \mathcal{T}_1(Z) \quad (6.8)$$

of the right Lie algebra \mathfrak{g}_r of G into the Lie algebra of vector fields on Z such that

$$\xi_{\text{Ad } g(\varepsilon)} = Tg \circ \xi_\varepsilon \circ g^{-1}. \quad (6.9)$$

Vector fields ξ_ε are said to be the infinitesimal generators of a representation of the Lie group G in Z .

In particular, the adjoint representation (6.7) of a Lie group in its right Lie algebra yields the adjoint representation

$$\varepsilon' : \varepsilon \rightarrow \text{ad } \varepsilon'(\varepsilon) = [\varepsilon', \varepsilon], \quad \text{ad } \varepsilon_m(\varepsilon_n) = c_{mn}^k \varepsilon_k,$$

of the right Lie algebra \mathfrak{g}_r in itself.

The dual $\mathfrak{g}^* = T_e^*G$ of the tangent space $T_e G$ is called the Lie coalgebra). It is provided with the basis $\{\varepsilon^m\}$ which is the dual of the basis $\{\varepsilon_m\}$ for $T_e G$. The group G and the right Lie algebra \mathfrak{g}_r act on \mathfrak{g}^* by the coadjoint representation

$$\begin{aligned} \langle \text{Ad}^* g(\varepsilon^*), \varepsilon \rangle &= \langle \varepsilon^*, \text{Ad } g^{-1}(\varepsilon) \rangle, \quad \varepsilon^* \in \mathfrak{g}^*, \quad \varepsilon \in \mathfrak{g}_r, \\ \langle \text{ad}^* \varepsilon'(\varepsilon^*), \varepsilon \rangle &= -\langle \varepsilon^*, [\varepsilon', \varepsilon] \rangle, \quad \varepsilon' \in \mathfrak{g}_r, \\ \text{ad}^* \varepsilon_m(\varepsilon^n) &= -c_{mn}^k \varepsilon^k. \end{aligned} \quad (6.10)$$

The Lie coalgebra \mathfrak{g}^* of a Lie group G is provided with the canonical Poisson structure, called the Lie–Poisson structure [1, 19]. It is given by the bracket

$$\{f, g\}_{\text{LP}} = \langle \varepsilon^*, [df(\varepsilon^*), dg(\varepsilon^*)] \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*), \quad (6.11)$$

where $df(\varepsilon^*), dg(\varepsilon^*) \in \mathfrak{g}_r$ are seen as linear mappings from $T_{\varepsilon^*} \mathfrak{g}^* = \mathfrak{g}^*$ to \mathbb{R} . Given coordinates z_k on \mathfrak{g}^* with respect to the basis $\{\varepsilon^k\}$, the Lie – Poisson bracket (6.11) and the corresponding Poisson bivector field w read

$$\{f, g\}_{\text{LP}} = c_{mn}^k z_k \partial^m f \partial^n g, \quad w_{mn} = c_{mn}^k z_k.$$

One can show that symplectic leaves of the Lie–Poisson structure on the coalgebra \mathfrak{g}^* of a connected Lie group G are orbits of the coadjoint representation (6.10) of G on \mathfrak{g}^* [32].

6.3 Symplectic structure

Let Z be a smooth manifold. Any exterior two-form Ω on Z yields a linear bundle morphism

$$\Omega^\flat : TZ \xrightarrow{Z} T^*Z, \quad \Omega^\flat : v \rightarrow -v \lrcorner \Omega(z), \quad v \in T_z Z, \quad z \in Z. \quad (6.12)$$

One says that a two-form Ω is of rank r if the morphism (6.12) has a rank r . A kernel $\text{Ker } \Omega$ of Ω is defined as the kernel of the morphism (6.12). In particular, $\text{Ker } \Omega$ contains the canonical zero section $\widehat{0}$ of $TZ \rightarrow Z$. If $\text{Ker } \Omega = \widehat{0}$, a two-form Ω is said to be non-degenerate. A closed non-degenerate two-form Ω is called symplectic. Accordingly, a manifold equipped with a symplectic form is a symplectic manifold. A symplectic manifold (Z, Ω) always is even dimensional and orientable.

A manifold morphism ζ of a symplectic manifold (Z, Ω) to a symplectic manifold (Z', Ω') is called symplectic if $\Omega = \zeta^* \Omega'$. Any symplectic morphism is an immersion. A symplectic isomorphism is called the symplectomorphism.

A vector field u on a symplectic manifold (Z, Ω) is an infinitesimal generator of a local one-parameter group of local symplectomorphism iff the Lie derivative $\mathbf{L}_u \Omega$ vanishes. It is called the canonical vector field. A canonical vector field u on a symplectic manifold (Z, Ω) is said to be Hamiltonian if a closed one-form $u \lrcorner \Omega$ is exact. Any smooth function $f \in C^\infty(Z)$ on Z defines a unique Hamiltonian vector field ϑ_f such that

$$\vartheta_f \lrcorner \Omega = -df, \quad \vartheta_f = \Omega^\sharp(df), \quad (6.13)$$

where Ω^\sharp is the inverse isomorphism to Ω^\flat (6.12).

Example 6.2: Given an m -dimensional manifold M coordinated by (q^i) , let

$$\pi_{*M} : T^*M \rightarrow M$$

be its cotangent bundle equipped with the holonomic coordinates $(q^i, p_i = \dot{q}_i)$. It is endowed with the canonical Liouville form

$$\Xi = p_i dq^i$$

and the canonical symplectic form

$$\Omega_T = d\Xi = dp_i \wedge dq^i. \quad (6.14)$$

Their coordinate expressions are maintained under holonomic coordinate transformations. The Hamiltonian vector field ϑ_f (6.13) with respect to the canonical symplectic form (6.14) reads

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.$$

□

The canonical symplectic form (6.14) plays a prominent role in symplectic geometry in view of the classical Darboux theorem.

THEOREM 6.4: Each point of a symplectic manifold (Z, Ω) has an open neighborhood equipped with coordinates (q^i, p_i) , called canonical or Darboux coordinates, such that Ω takes the coordinate form (6.14). \square

Let $i_N : N \rightarrow Z$ be a submanifold of a $2m$ -dimensional symplectic manifold (Z, Ω) . A subset

$$\text{Orth}_\Omega TN = \bigcup_{z \in N} \{v \in T_z Z : v \lrcorner \Omega = 0, u \in T_z N\}$$

of $TZ|_N$ is called orthogonal to TN relative to a symplectic form Ω . One considers the following special types of submanifolds of a symplectic manifold such that the pull-back $\Omega_N = i_N^* \Omega$ of a symplectic form Ω onto a submanifold N is of constant rank. A submanifold N of Z is said to be:

- coisotropic if $\text{Orth}_\Omega TN \subseteq TN$, $\dim N \geq m$;
- symplectic if Ω_N is a symplectic form on N ;
- isotropic if $TN \subseteq \text{Orth}_\Omega TN$, $\dim N \leq m$.

6.4 Poisson structure

A Poisson bracket on a ring $C^\infty(Z)$ of smooth real functions on a manifold Z (or a Poisson structure on Z) is defined as an \mathbb{R} -bilinear map

$$C^\infty(Z) \times C^\infty(Z) \ni (f, g) \rightarrow \{f, g\} \in C^\infty(Z)$$

which satisfies the following conditions:

- $\{g, f\} = -\{f, g\}$;
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$;
- $\{h, fg\} = \{h, f\}g + f\{h, g\}$.

A Poisson bracket makes $C^\infty(Z)$ into a real Lie algebra, called the Poisson algebra. A Poisson structure is characterized by a particular bivector field as follows.

THEOREM 6.5: Every Poisson bracket on a manifold Z is uniquely defined as

$$\{f, f'\} = w(df, df') = w^{\mu\nu} \partial_\mu f \partial_\nu f' \quad (6.15)$$

by a bivector field w whose Schouten–Nijenhuis bracket $[w, w]_{\text{SN}}$ vanishes. It is called a Poisson bivector field. \square

A manifold Z endowed with a Poisson structure is called a Poisson manifold.

Example 6.3: Any manifold admits a zero Poisson structure characterized by a zero Poisson bivector field $w = 0$. \square

Any bivector field w on a manifold Z yields a linear bundle morphism

$$w^\sharp : T^*Z \xrightarrow{Z} TZ, \quad w^\sharp : \alpha \rightarrow -w(z) \lrcorner \alpha, \quad \alpha \in T_z^*Z. \quad (6.16)$$

One says that w is of rank r if the morphism (6.16) is of this rank. If a Poisson bivector field is of constant rank, the Poisson structure is called regular. Throughout this work, only regular Poisson structures are considered. A Poisson structure determined by a Poisson bivector field w is said to be non-degenerate if w is of maximal rank.

There is one-to-one correspondence $\Omega_w \leftrightarrow w_\Omega$ between the symplectic forms and the non-degenerate Poisson bivector fields which is given by the equalities

$$\begin{aligned} w_\Omega(\phi, \sigma) &= \Omega_w(w_\Omega^\sharp(\phi), w_\Omega^\sharp(\sigma)), & \phi, \sigma \in \mathcal{O}^1(Z), \\ \Omega_w(\vartheta, \nu) &= w_\Omega(\Omega_w^\flat(\vartheta), \Omega_w^\flat(\nu)), & \vartheta, \nu \in \mathcal{T}(Z), \end{aligned}$$

where the morphisms w_Ω^\sharp (6.16) and Ω_w^\flat (6.12) are mutually inverse.

However, this correspondence is not preserved under manifold morphisms in general. Namely, let (Z_1, w_1) and (Z_2, w_2) be Poisson manifolds. A manifold morphism $\varrho : Z_1 \rightarrow Z_2$ is said to be a Poisson morphism if

$$\{f \circ \varrho, f' \circ \varrho\}_1 = \{f, f'\}_2 \circ \varrho, \quad f, f' \in C^\infty(Z_2),$$

or, equivalently, if $w_2 = T\varrho \circ w_1$, where $T\varrho$ is the tangent map to ϱ . Herewith, the rank of w_1 is superior or equal to that of w_2 . Therefore, there are no pull-back and push-forward operations of Poisson structures in general. Nevertheless, let us mention the following construction.

THEOREM 6.6: Let (Z, w) be a Poisson manifold and $\pi : Z \rightarrow Y$ a fibration such that, for every pair of functions (f, g) on Y and for each point $y \in Y$, the restriction of a function $\{\pi^*f, \pi^*g\}$ to a fibre $\pi^{-1}(y)$ is constant, i.e., $\{\pi^*f, \pi^*g\}$ is the pull-back onto Z of some function on Y . Then there exists a coinduced Poisson structure w' on Y for which π is a Poisson morphism. \square

Example 6.4: The direct product $Z \times Z'$ of Poisson manifolds (Z, w) and (Z', w') can be endowed with the product of Poisson structures, given by a bivector field $w + w'$ such that the surjections pr_1 and pr_2 are Poisson morphisms. \square

A function $f \in C^\infty(Z)$ is called the Casimir function of a Poisson structure on Z if its Poisson bracket with any function on Z vanishes. Casimir functions form a real ring $\mathcal{C}(Z)$. In particular, a symplectic manifold admits only constant Casimir functions.

A vector field u on a Poisson manifold (Z, w) is an infinitesimal generator of a local one-parameter group of Poisson automorphisms iff the Lie derivative

$$\mathbf{L}_u w = [u, w]_{\text{SN}} \quad (6.17)$$

vanishes. It is called the canonical vector field for a Poisson structure w . In particular, for any real smooth function f on a Poisson manifold (Z, w) , let us put

$$\vartheta_f = w^\sharp(df) = -w \lrcorner df = w^{\mu\nu} \partial_\mu f \partial_\nu. \quad (6.18)$$

It is a canonical vector field, called the Hamiltonian vector field of a function f with respect to a Poisson structure w . Hamiltonian vector fields fulfil the relations

$$\{f, g\} = \vartheta_f \lrcorner dg, \quad [\vartheta_f, \vartheta_g] = \vartheta_{\{f, g\}}, \quad f, g \in C^\infty(Z). \quad (6.19)$$

For instance, the Hamiltonian vector field ϑ_f (6.13) of a function f on a symplectic manifold (Z, Ω) coincides with that (6.18) with respect to the corresponding Poisson structure w_Ω . The Poisson bracket defined by a symplectic form Ω reads

$$\{f, g\} = \vartheta_g \lrcorner \vartheta_f \lrcorner \Omega.$$

Since a Poisson manifold (Z, w) is assumed to be regular, the range $\mathbf{T} = w^\#(T^*Z)$ of the morphism (6.16) is a subbundle of TZ called the characteristic distribution on (Z, w) . It is spanned by Hamiltonian vector fields, and it is involutive by virtue of the relation (6.19). It follows that a Poisson manifold Z admits local adapted coordinates in Theorem 6.1. Moreover, one can choose particular adapted coordinates which bring a Poisson structure into the following canonical form.

THEOREM 6.7: For any point z of a k -dimensional Poisson manifold (Z, w) , there exist coordinates

$$(z^1, \dots, z^{k-2m}, q^1, \dots, q^m, p_1, \dots, p_m) \quad (6.20)$$

on a neighborhood of z such that

$$w = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, \quad \{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}.$$

□

The coordinates (6.20) are called the canonical or Darboux coordinates for the Poisson structure w . The Hamiltonian vector field of a function f written in this coordinates is

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.$$

Of course, the canonical coordinates for a symplectic form Ω in Theorem 6.4 also are canonical coordinates in Theorem 6.7 for the corresponding non-degenerate Poisson bivector field w , i.e.,

$$\Omega = dp_i \wedge dq^i, \quad w = \partial^i \wedge \partial_i.$$

With respect to these coordinates, the mutually inverse bundle isomorphisms Ω^\flat (6.12) and $w^\#$ (6.16) read

$$\begin{aligned} \Omega^\flat : v^i \partial_i + v_i \partial^i &\rightarrow -v_i dq^i + v^i dp_i, \\ w^\# : v_i dq^i + v^i dp_i &\rightarrow v^i \partial_i - v_i \partial^i. \end{aligned}$$

6.5 Symplectic foliations

Integral manifolds of the characteristic distribution \mathbf{T} of a k -dimensional Poisson manifold (Z, w) constitute a (regular) foliation \mathcal{F} of Z whose tangent bundle $T\mathcal{F}$ is \mathbf{T} . It is called the characteristic foliation of a Poisson manifold. By the very definition of the characteristic distribution $\mathbf{T} = T\mathcal{F}$, a Poisson bivector field w is subordinate to $\overset{2}{\wedge} T\mathcal{F}$. Therefore, its restriction $w|_F$ to any leaf F of \mathcal{F} is a non-degenerate Poisson bivector field on F . It provides F with a non-degenerate Poisson structure $\{\cdot, \cdot\}_F$ and, consequently, a symplectic structure. Clearly, the local Darboux coordinates for the Poisson structure w in Theorem 6.7 also are the local adapted coordinates

$$(z^1, \dots, z^{k-2m}, z^i = q^i, z^{m+i} = p_i), \quad i = 1, \dots, m,$$

(6.1) for the characteristic foliation \mathcal{F} , and the symplectic structures along its leaves read

$$\Omega_F = dp_i \wedge dq^i.$$

In particular, it follows that Casimir functions of a Poisson structure are constant on leaves of its characteristic symplectic foliation.

Since any foliation is locally simple, a local structure of an arbitrary Poisson manifold reduces to the following [29, 32].

THEOREM 6.8: Each point of a Poisson manifold has an open neighborhood which is Poisson equivalent to the product of a manifold with the zero Poisson structure and a symplectic manifold. \square

Provided with this symplectic structure, the leaves of the characteristic foliation of a Poisson manifold Z are assembled into a symplectic foliation of Z as follows.

Let \mathcal{F} be an even dimensional foliation of a manifold Z . A \tilde{d} -closed non-degenerate leafwise two-form $\Omega_{\mathcal{F}}$ on a foliated manifold (Z, \mathcal{F}) is called symplectic. Its pull-back $i_F^* \Omega_{\mathcal{F}}$ onto each leaf F of \mathcal{F} is a symplectic form on F . A foliation \mathcal{F} provided with a symplectic leafwise form $\Omega_{\mathcal{F}}$ is called the symplectic foliation.

If a symplectic leafwise form $\Omega_{\mathcal{F}}$ exists, it yields a bundle isomorphism

$$\Omega_{\mathcal{F}}^b : T\mathcal{F} \xrightarrow{Z} T\mathcal{F}^*, \quad \Omega_{\mathcal{F}}^b : v \rightarrow -v \rfloor \Omega_{\mathcal{F}}(z), \quad v \in T_z \mathcal{F}.$$

The inverse isomorphism $\Omega_{\mathcal{F}}^\sharp$ determines a bivector field

$$w_\Omega(\alpha, \beta) = \Omega_{\mathcal{F}}(\Omega_{\mathcal{F}}^\sharp(i_{\mathcal{F}}^* \alpha), \Omega_{\mathcal{F}}^\sharp(i_{\mathcal{F}}^* \beta)), \quad \alpha, \beta \in T_z^* Z, \quad z \in Z, \quad (6.21)$$

on Z subordinate to $\overset{2}{\wedge} T\mathcal{F}$. It is a Poisson bivector field. The corresponding Poisson bracket reads

$$\{f, f'\}_{\mathcal{F}} = \vartheta_f \rfloor \tilde{d}f', \quad \vartheta_f \rfloor \Omega_{\mathcal{F}} = -\tilde{d}f, \quad \vartheta_f = \Omega_{\mathcal{F}}^\sharp(\tilde{d}f). \quad (6.22)$$

Its kernel is $S_{\mathcal{F}}(Z)$.

Conversely, let (Z, w) be a Poisson manifold and \mathcal{F} its characteristic foliation. Since $\text{Ann } T\mathcal{F} \subset T^*Z$ is precisely the kernel of a Poisson bivector field w , a bundle homomorphism

$$w^\sharp : T^*Z \xrightarrow[Z]{} TZ$$

factorizes in a unique fashion

$$w^\sharp : T^*Z \xrightarrow[Z]{i_{\mathcal{F}}^*} T\mathcal{F}^* \xrightarrow[Z]{w_{\mathcal{F}}^\sharp} T\mathcal{F} \xrightarrow[Z]{i_{\mathcal{F}}} TZ \quad (6.23)$$

through a bundle isomorphism

$$w_{\mathcal{F}}^\sharp : T\mathcal{F}^* \xrightarrow[Z]{} T\mathcal{F}, \quad w_{\mathcal{F}}^\sharp : \alpha \rightarrow -w(z)|\alpha, \quad \alpha \in T_z\mathcal{F}^*. \quad (6.24)$$

The inverse isomorphism $w_{\mathcal{F}}^\flat$ yields a symplectic leafwise form

$$\Omega_{\mathcal{F}}(v, v') = w(w_{\mathcal{F}}^\flat(v), w_{\mathcal{F}}^\flat(v')), \quad v, v' \in T_z\mathcal{F}, \quad z \in Z. \quad (6.25)$$

The formulas (6.21) and (6.25) establish the equivalence between the Poisson structures on a manifold Z and its symplectic foliations.

6.6 Group action on Poisson manifolds

Turn now to a group action on Poisson manifolds. By G throughout is meant a real connected Lie group, \mathfrak{g} is its right Lie algebra, and \mathfrak{g}^* is the Lie coalgebra (see Section 7.5).

We start with the symplectic case. Let a Lie group G act on a symplectic manifold (Z, Ω) on the left by symplectomorphisms. Such an action of G is called symplectic. Since G is connected, its action on a manifold Z is symplectic iff the homomorphism $\varepsilon \rightarrow \xi_\varepsilon$, $\varepsilon \in \mathfrak{g}$, (6.8) of a Lie algebra \mathfrak{g} to a Lie algebra $\mathcal{T}_1(Z)$ of vector fields on Z is carried out by canonical vector fields for a symplectic form Ω on Z . If all these vector fields are Hamiltonian, an action of G on Z is called a Hamiltonian action. One can show that, in this case, ξ_ε , $\varepsilon \in \mathfrak{g}$, are Hamiltonian vector fields of functions on Z of the following particular type.

PROPOSITION 6.9: An action of a Lie group G on a symplectic manifold Z is Hamiltonian iff there exists a mapping

$$\widehat{J} : Z \rightarrow \mathfrak{g}^*, \quad (6.26)$$

called the momentum mapping, such that

$$\xi_\varepsilon \lrcorner \Omega = -dJ_\varepsilon, \quad J_\varepsilon(z) = \langle \widehat{J}(z), \varepsilon \rangle, \quad \varepsilon \in \mathfrak{g}. \quad (6.27)$$

□

The momentum mapping (6.26) is defined up to a constant map. Indeed, if \hat{J} and \hat{J}' are different momentum mappings for the same symplectic action of G on Z , then

$$d(\langle \hat{J}(z) - \hat{J}'(z), \varepsilon \rangle) = 0, \quad \varepsilon \in \mathfrak{g}.$$

Given $g \in G$, let us consider the difference

$$\sigma(g) = \hat{J}(gz) - \text{Ad}^*g(\hat{J}(z)), \quad (6.28)$$

where Ad^*g is the coadjoint representation (6.10) on \mathfrak{g}^* . One can show that the difference (6.28) is constant on a symplectic manifold Z [1]. A momentum mapping \hat{J} is called equivariant if $\sigma(g) = 0$, $g \in G$.

Example 6.5: Let a symplectic form on Z be exact, i.e., $\Omega = d\theta$, and let θ be G -invariant, i.e.,

$$\mathbf{L}_{\xi_\varepsilon}\theta = d(\xi_\varepsilon \lrcorner \theta) + \xi_\varepsilon \lrcorner \Omega = 0, \quad \varepsilon \in \mathfrak{g}.$$

Then the momentum mapping \hat{J} (6.26) can be given by the relation

$$\langle \hat{J}(z), \varepsilon \rangle = (\xi_\varepsilon \lrcorner \theta)(z).$$

It is equivariant. In accordance with the relation (6.10), it suffices to show that

$$J_\varepsilon(gz) = J_{\text{Ad } g^{-1}(\varepsilon)}(z), \quad (\xi_\varepsilon \lrcorner \theta)(gz) = (\xi_{\text{Ad } g^{-1}(\varepsilon)} \lrcorner \theta)(z).$$

This holds by virtue of the relation (6.9). For instance, let T^*Q be a symplectic manifold equipped with the canonical symplectic form Ω_T (6.14). Let a left action of a Lie group G on Q have the infinitesimal generators $\tau_m = \varepsilon_m^i(q)\partial_i$. The canonical lift of this action onto T^*Q has the infinitesimal generators

$$\xi_m = \tilde{\tau}_m = v e_m^i \partial_i - p_j \partial_i \varepsilon_m^j \partial^i, \quad (6.29)$$

and preserves the canonical Liouville form Ξ on T^*Q . The ξ_m (6.29) are Hamiltonian vector fields of the functions $J_m = \varepsilon_m^i(q)p_i$, determined by the equivariant momentum mapping $\hat{J} = \varepsilon_m^i(q)p_i \varepsilon^m$. \square

THEOREM 6.10: A momentum mapping \hat{J} associated to a symplectic action of a Lie group G on a symplectic manifold Z obeys the relation

$$\{J_\varepsilon, J_{\varepsilon'}\} = J_{[\varepsilon, \varepsilon']} - \langle T_e \sigma(\varepsilon'), \varepsilon \rangle. \quad (6.30)$$

\square

In the case of an equivariant momentum mapping, the relation (6.30) leads to a homomorphism

$$\{J_\varepsilon, J_{\varepsilon'}\} = J_{[\varepsilon, \varepsilon']} \quad (6.31)$$

of a Lie algebra \mathfrak{g} to a Poisson algebra of smooth functions on a symplectic manifold Z (cf. Proposition 6.11 below).

Now let a Lie group G act on a Poisson manifold (Z, w) on the left by Poisson automorphism. This is a Poisson action. Since G is connected, its action on a manifold Z is a Poisson action iff the homomorphism $\varepsilon \rightarrow \xi_\varepsilon$, $\varepsilon \in \mathfrak{g}$, (6.8) of a Lie algebra \mathfrak{g} to a Lie algebra $\mathcal{T}_1(Z)$ of vector fields on Z is carried out by canonical vector fields for a Poisson bivector field w , i.e., the condition (6.17) holds. The equivalent conditions are

$$\begin{aligned}\xi_\varepsilon(\{f, g\}) &= \{\xi_\varepsilon(f), g\} + \{f, \xi_\varepsilon(g)\}, & f, g &\in C^\infty(Z), \\ \xi_\varepsilon(\{f, g\}) &= [\xi_\varepsilon, \vartheta_f](g) - [\xi_\varepsilon, \vartheta_g](f), \\ [\xi_\varepsilon, \vartheta_f] &= \vartheta_{\xi_\varepsilon(f)},\end{aligned}$$

where ϑ_f is the Hamiltonian vector field (6.18) of a function f .

A Hamiltonian action of G on a Poisson manifold Z is defined similarly to that on a symplectic manifold. Its infinitesimal generators are tangent to leaves of the symplectic foliation of Z , and there is a Hamiltonian action of G on every symplectic leaf. Proposition 6.9 together with the notions of a momentum mapping and an equivariant momentum mapping also are extended to a Poisson action. However, the difference σ (6.28) is constant only on leaves of the symplectic foliation of Z in general. At the same time, one can say something more on an equivariant momentum mapping (that also is valid for a symplectic action).

PROPOSITION 6.11: An equivariant momentum mapping \widehat{J} (6.26) is a Poisson morphism to the Lie coalgebra \mathfrak{g}^* , provided with the Lie–Poisson structure (6.11). \square

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